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Free-Boundary Regularity for a Problem Arising in Superconductivity

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Abstract

This paper concerns regularity properties of the mean-field theory of superconductivity. The problem is reminiscent of the one studied earlier by L.A. Caffarelli, L. Karp and H. Shahgholian in connection with potential theory. The difficulty introduced in this paper is the existence of several patches, where on each patch the solution to the problem may have different constant values. However, using a refined analysis, we reduce the problem to the one-patch case, at least locally near "regular" free boundary points. Using a monotonicity formula, due to Georg S. Weiss, we characterize global solutions of a related equation. Hence earlier regularity results apply and we conclude the C^1 regularity of the free boundary.

1. Introduction

In analyzing the evolution of vortices arising in the mean-field model of penetration of the magnetic field into super-conducting bodies, we end up with a degenerate parabolic-elliptic system (see [8] for details). A simplified stationary model of this problem (in a local setting), where the scalar stream function admits a functional dependence on the scalar magnetic potential, reduces to finding u such that

$$\Delta u = u\chi_{\{|\nabla u|>0\}}, \quad u \geqq 0, \quad \text{in } B_\rho(x_0), \tag{1}$$

where $B_\rho(x_0)$ denotes the ball of radius ρ centered at $\zeta \in \mathbf{R}^n$), the equation is in the sense of distribution, and appropriate boundary data are fulfilled.

Related problems have been studied in [1, 3, 8, 9]; see also the references therein. However, less attention has been paid to the regularity nature of the solution function u and the free boundary $\partial\{|\nabla u| > 0\}$.

Existence of solutions of the Dirichlet problem associated with this equation was studied in [7], where the $C^{1,1}$ interior regularity and the local finiteness of the $(n-1)$-dimensional Hausdorff measure of the free boundary was established.

In this paper, it is our prime goal to analyze the above problem in the context of regularity theory. Using a refined analysis inspired by techniques introduced in [6], we reduce the problem to the one-patch case, near "regular" free boundary points.

2. Definitions and known results

In most of the paper, we deal with functions $u \in C^{1,1}(B_\rho(x_0))$, $0 < \rho \leqq \infty$, $x_0 \in \mathbf{R}^n$, which satisfy the differential equation

$$\Delta u = \chi_{\{|\nabla u|>0\}}, \tag{2}$$

with

$$|\nabla u(x_0)| = 0\,, \quad |u(x) - u(x_0)| \leqq c(1 + |x|^2), \tag{3}$$

where c is a fixed positive constant. The differential equation (2) is interpreted in the sense of distributions.

Notation. We denote by $\mathcal{P}(c, \rho, x_0)$ the class of functions $u \in C^{1,1}(B_\rho(x_0))$ verifying (2) and (3). Global solutions to (2), (3) are denoted by $\mathcal{P}$, i.e.,

$$\mathcal{P} := \bigcup_{x_0 \in \mathbf{R}^n} \bigcup_{c>0} \bigcap_{\rho>0} \mathcal{P}(c, \rho, x_0).$$

The most relevant result we use from the regularity theory developed in [6] and [7], is the following theorem.

Theorem 1 ([6] and [7])**.** *For $u \in \mathcal{P}(c, \rho, x_0)$, the following uniform $C^{1,1}$ estimate holds:*

$$\sup_{B_{\rho/2} \cap \{|\nabla u|>0\}} |D_{ij} u(x)| \leqq C,$$

where C is a constant depending only on c and n.

Moreover, the free boundary $\partial\{|\nabla u| > 0\}$ has locally finite $(n-1)$-Hausdorff measure.

The proof given in [7] should be slightly changed (cf. [6]) so that it holds for the whole class. However, the proof given in [6] works perfectly in this case.

Definition 1. By Theorem 1, for all $u \in \mathcal{P}(c, \rho, x_0)$, $\Delta u = 1$, in the classical sense, in the set

$$\Omega := \operatorname{int}\overline{\{|\nabla u| > 0\}}.$$

Remark 1. If $u \in \mathcal{P}(c, \rho, x_0)$ then necessarily $|\nabla u(x_0)| = 0$, even though x_0 might not be in $\Omega^c := \mathbf{R}^n \setminus \Omega$. If $u \in \mathcal{P}$ is a global solution, x_0 does not need to be the origin and it can be translated since the equation is translation invariant. As a matter of fact, in Section 6, the origin is carefully chosen using Lemma 7.

3. Regularity of the free boundary

Before establishing our main result, we need the following definition.

Definition 2. The *minimal diameter* of a bounded set $D \subset \mathbf{R}^n$, denoted $\mathrm{MD}(D)$, is the infimum of distances between pairs of parallel planes such that D is contained in the strip determined by the planes.

Definition 3. The *density function* of Ω^c (at the origin) is defined by

$$\delta_\rho(u) = \frac{\mathrm{MD}(\Omega^c \cap B_\rho)}{\rho}.$$

Theorem 2. *There exists a modulus of continuity σ such that if $\delta_{\rho_0}(u) > \sigma(\rho_0)$ for some $\rho_0 < 1/2$, then for any $u \in \mathcal{P}(c, 1, x_0)$, the boundary $\partial\{|\nabla u| > 0\}$ is a C^1 graph in $B(x_0, c_0{\rho_0}^2)$. Here c_0 is a universal constant, depending only on c, and the dimension n.*

This theorem will be a consequence of the lemmas presented in the following sections, combined with Theorem III in [6], and the main result in [4]. To apply Theorem 2 to solutions of (1), we need the following lemma.

Lemma 1 ([7], Lemma 10)**.** *Assume that u is a non-negative solution to* (1) *and $x_0 \in \partial\{|\nabla u| > 0\}$. Then $u > 0$.*

Although in Theorem 2 we consider the case of a constant right-hand side (constant in the set $\{|\nabla u| > 0\}$), our analysis works perfectly for solutions of equation (1). The constant c_0 in the theorem will then depend on $u(x_0)$ as well. We leave the small changes needed in this case to the reader, and continue our analysis in the rest of this paper for the case shown in equation (2).

4. Further definitions and preliminary results

Given a function $u \in \mathcal{P}(c, \rho, x_0)$, let us define the r-scaled function of u at x_0 as

$$u_r(x) := \frac{u(x_0 + rx) - u(x_0)}{r^2}, \qquad 0 < r \leqq \rho,\ x \in B_1(0).$$

Set also

$$\Omega_r = \left\{x \in \mathbf{R}^n ;\ x_0 + rx \in \Omega\right\}.$$

By Theorem 1, $\{u_r\}_{0<r\leqq\rho}$ is a relatively compact family. By the Ascoli-Arzela theorem, given any sequence of positive numbers $\{r\}$ tending to zero, there is a subsequence $r_k \to 0$ such that u_{r_k} converges uniformly on compact sets to a globally defined function

$$u_0 = \lim_{k\to\infty} u_{r_k} \in \mathcal{P}.$$

We refer to this function as blow-up of u at x_0 (with respect to the sequence $\{r_k\}$).

Although blow-ups at a fixed point $x_0 \in \mathbf{R}^n$ might depend *a priori* on the sequence $r_k \to 0$, we denote any blow-up at x_0 by u_0. This will cause no confusion since we do not use more than one blow-up at the same time.

If $u \in \mathcal{P}$, u_r is defined for all $r > 0$, since the family $\{u_r\}_{r>0}$ is also relatively compact, we may consider blowing up at ∞. We denote any blow-up at ∞ by u_∞. As above, this will cause no confusion and eventually $u_r \to u_\infty$ when $r \to \infty$.

Lemma 2. *If $u \in \mathcal{P}(c, \rho, x_0)$, then any blow-up u_0 at a free boundary point is either:*

– *a half-space solution, i.e.,*

$$u_0(x) = \tfrac{1}{2}[(x \cdot \nu_0)^+]^2,$$

for some $\nu_0 \in S^{n-1}$, or

– *a Homogeneous, degree-two polynomial $P(x)$ with $\Delta P = 1$.*

Remark 2. Due to the non-degeneracy of solutions (see [7]), i.e.,

$$\sup_{B_r(y)} u \geqq \gamma r^2 + u(y), \qquad y \in \{|\nabla u| > 0\},$$

blow-ups do not vanish identically. Here the constant γ depends only on the space dimension.

Proof (of Lemma 2)**.** We apply the monotonicity formula of ALT, CAFFARELLI & FRIEDMAN [2] to the positive and negative parts of a directional derivative of u. To fix the notation, set

$$\varphi(r, \nu, u) = \varphi(r, x_0, \nu, u) = \frac{1}{r^4} \int_{B_r(x_0)} \frac{\left|\nabla (D_\nu u)^+\right|^2}{|x - x_0|^{n-2}} \int_{B_r(x_0)} \frac{\left|\nabla (D_\nu u)^-\right|^2}{|x - x_0|^{n-2}},$$

where $r > 0$ and $(D_\nu u)^\pm$ denotes the positive and negative parts (respectively) of the directional derivative of u in the direction $\nu \in S^{n-1}$. (In what follows we skip the x_0 dependents of φ.) Fix a sequence $r_k \to 0$ such that

$$u_0 = \lim_{k \to \infty} u_{r_k}.$$

Since u_{r_k} converges in $W^{2,p}$,

$$\varphi(s, \nu, u_0) = \lim_{k \to \infty} \varphi(s, \nu, u_{r_k}).$$

Using a change of variable we readily verify that

$$\varphi(1, \nu, u_r) = \varphi(r, \nu, u).$$

This and the monotonicity lemma (Lemma 5.1 in [2]; cf. also [6]), which says that $\varphi(r, \nu, u)$ is monotone with respect to r, give

$$\varphi(s, \nu, u_0) = \lim_{k \to \infty} \varphi(s r_k, \nu, u) := \varphi(0^+, \nu, u).$$

Hence, for any blow-up u_0, $\varphi(r, \nu, u_0)$ is constant with respect to r and

$$\varphi(r, \nu, u_0) = \varphi(0^+, \nu, u) \leqq \varphi(r, \nu, u).$$

Remark 3 ([6])**.** Given $v \in \mathcal{P}$,

$$\varphi(r, \nu, v) = 0, \quad \forall r > 0, \ \forall \nu \in S^{n-1}$$

if and only if, for some $a \in \mathbf{R}$, $v - a$ is a half-space solution.

Therefore if u_0 is not a half-space solution, there is $\nu \in S^{n-1}$ such that $\varphi(r, \nu, u_0) > 0$. By the strong form of the monotonicity formula (see Lemma 2.2 in [6]), $\Omega^c = \emptyset$. Applying Liouville's theorem to the second-order partial derivatives of u_0, we see that u_0 is a polynomial of degree two; the homogeneity comes from the fact that $u_0(0) = |\nabla u_0(0)| = 0$. □

In the same way we can prove the following lemma.

Lemma 3. *If $u \in \mathcal{P}$, then any blow-up u_∞ at ∞ is either: a half-space solution or a homogeneous, degree-two polynomial.*

Using Remark 3 and the formula

$$\varphi(r, \nu, u_0) = \varphi(0^+, \nu, u) \leqq \varphi(r, \nu, u) \leqq \varphi(\infty^-, \nu, u) = \varphi(r, \nu, u_\infty),$$

we can conclude as in the next proposition.

Proposition 1. *For functions $u \in \mathcal{P}$, the following hold.*

– *A blow-up u_0 at x_0 ($\in \partial\Omega$) is a half-space solution if and only if*

$$\varphi(0^+, \nu, u) = 0, \quad \forall \nu \in S^{n-1}.$$

– *If some blow-up u_0 at the origin is a half-space solution, then any blow-up at the origin is a half-space solution.*
– *If some blow-up u_∞ of u at ∞ is a half-space solution, then $u - u(x_0)$ is a half-space solution or a translation of one, $x_0 \in \Omega^c$.*

5. The local structure of the patches

In this section we will gather some technical lemmas that will be referred to, quite frequently, in the rest of the paper.

First we need a definition. For $\rho > 0$, $0 < \delta < 1$, $\nu \in S^{n-1}$, we define

$$C(\rho, \delta, \nu) := \left\{ x \in \mathbf{R}^n;\ 0 < |x| \leqq \rho\,,\ \frac{x}{|x|} \cdot \nu \leqq -1 + \delta \right\}.$$

Lemma 4. *Assume that $u \in \mathcal{P}(c, 1, x_0)$ and there exists a blow-up u_0 at x_0 that is a half-space solution, i.e.,*

$$u_0(x) = \tfrac{1}{2}[(x \cdot \nu_0)^+]^2.$$

Then, there is $\rho > 0$ such that for all $\nu \in C(1, \frac{1}{2}, -\nu_0) \cap S^{n-1}$,

$$2D_\nu u - |\nabla u|^2 \geqq 0 \quad \text{in } B_\rho(x_0).$$

In particular, u is non-decreasing in $B_\rho(x_0)$, in the direction of ν, for all $\nu \in C(1, \frac{1}{2}, -\nu_0) \cap S^{n-1}$.

Proof. It follows from the hypothesis that

$$D_\nu u_0 = (x \cdot \nu_0)^+ \nu_0 \cdot \nu \;\text{ and }\; |\nabla u_0|^2 = [(x \cdot \nu_0)^+]^2.$$

Hence, for all $\nu \in C(1, \frac{1}{2}, -\nu_0) \cap S^{n-1}$,

$$2D_\nu u_0 - |\nabla u_0|^2 \geqq 0 \quad \text{in } B_1 .$$

Fix a sequence $r_k \to 0$ such that

$$u_0 = \lim_{k\to\infty} u_{r_k} .$$

By uniform convergence, for all $\varepsilon > 0$, there is $k_0 \in \mathbb{N}$ such that, for all $k \geqq k_0$ and $\nu \in C(1, \frac{1}{2}, -\nu_0) \cap S^{n-1}$, we have

$$2D_\nu u_{r_k} - |\nabla u_{r_k}|^2 \geqq -\varepsilon \quad \text{in } B_1 .$$

Since

$$-\Delta(2D_\nu u_{r_k} - |\nabla u_{r_k}|^2) = 2|\nabla^2 u_{r_k}|^2 \geqq \frac{2}{n^2} \quad \text{in } B_1 \cap \Omega_{r_k} ,$$

we are in a position to apply the same argument as the one in the proof of Lemma 4.2 in [6] to conclude that there is a universal constant ε_0 (independent of u_{r_k} and ν) such that $\varepsilon \leqq \varepsilon_0$ implies

$$2D_\nu u_{r_k} - |\nabla u_{r_k}|^2 \geqq 0 , \quad \forall x \in B_{\frac{1}{2}} ,\ \forall \nu \in C(1, \tfrac{1}{2}, -\nu_0) \cap S^{n-1},$$

as soon as we choose k big enough. Taking $\rho = \frac{r_k}{2}$, the lemma is proved. □

Lemma 5. *Under the assumptions and notation of Lemma 4, for ρ as in Lemma 4 we have*

$$x_0 + C(\rho, \tfrac{1}{2}, \nu_0) \subset \left\{u \leqq u(x_0)\right\} .$$

Proof. For all (fixed)

$$x \in C(\rho, \tfrac{1}{2}, \nu_0),$$

the function

$$\tau \to u\,(x_0 + (1-\tau)x) \qquad \tau \in (0,1)$$

is non-decreasing. Indeed,

$$\frac{\partial}{\partial\tau}(u(x_0 + (1-\tau)x)) = -x \cdot \nabla u(x_0 + (1-\tau)x) = |x| D_\nu u(x_0 + (1-\tau)x),$$

where

$$\nu = \frac{-x}{|x|} \in C(1, \tfrac{1}{2}, -\nu_0) \cap S^{n-1}.$$

By Lemma 4, the derivative is non-negative and the lemma is proved. □

Lemma 6. *Under the assumptions and notation of Lemma 4, there is $\rho' > 0$ ($\rho' = \rho'(\rho)$) such that*

$$x_0 + C(\rho', \tfrac{1}{3}, \nu_0) \subset \Omega^c.$$

Proof. Suppose there is a sequence $\{x_k\} \subset C(1, \frac{1}{3}, \nu_0) \cap \Omega$ such that $\rho_k = |x_k| \to 0$. Fix a constant $\tau > 0$, such that for all $k \in \mathbb{N}$,

$$B_{\tau\rho_k}(x_k) = \left\{ y;\ |y - x_k| \leqq \tau\rho_k \right\} \subset C(1, \tfrac{1}{2}, \nu_0).$$

By the quadratic growth of u in Ω, there is a sequence $\{y_k\}$ such that

$$y_k \in B_{\tau\rho_k}(x_k) \qquad \forall k,$$

and

$$u_{\rho_k}({\rho_k}^{-1}y_k) - u_{\rho_k}({\rho_k}^{-1}x_k) \geqq \gamma \tag{4}$$

for some constant $\gamma > 0$, independent of k.

Now, by Proposition 1, any blow-up of u at x_0 is a half-space solution. In particular

$$\liminf_{r\to 0} u_r(x) \geqq 0 \qquad \forall x \in \mathbf{R}^n.$$

On the other hand, by Lemma 5,

$$\limsup_{r\to 0} u_r(x) \leqq 0\,, \qquad \forall x \in C(\infty, \tfrac{1}{2}, \nu_0).$$

Hence, for all $x \in C(\infty, \frac{1}{2}, \nu_0)$,

$$\lim_{r\to 0} u_r(x) = 0.$$

Since $\{{\rho_k}^{-1}y_k\}$ and $\{{\rho_k}^{-1}x_k\}$ are two bounded sequences contained in $C(2, \frac{1}{2}, \nu_0)$,

$$\lim_{k\to\infty} u_{\rho_k}({\rho_k}^{-1}y_k) - u_{\rho_k}({\rho_k}^{-1}x_k) = 0,$$

which is a contradiction to (4). $\square$

6. Global solutions with compact Ω^c

In this and the next section, we characterize global solutions. This characterization is useful for a further study of convexity properties. It is shown that any global solution to (2), with quadratic growth, either solves

$$\Delta u = \chi_{\{u>a\}} \quad \text{in } \mathbf{R}^n, \qquad u \geqq a, \tag{5}$$

for some $a \in \mathbf{R}$, or it is a degree-two polynomial.

Equation (5) was treated by CAFFARELLI in relation with the obstacle problem; see [5] and the reference there. The above shows that, as far as global solutions are concerned, (2) reduces to the one-patch problem treated in [6]: Given $u \in \mathcal{P}$, there is $a \in \mathbf{R}$, such that

$$\Delta u = \chi_\Omega \quad \text{in } \mathbf{R}^n$$

$$u = a\,,\ |\nabla u| = 0 \quad \text{in } \Omega^c.$$

Theorem 3. *Let $u \in \mathcal{P}$ and assume that Ω^c is non-empty and compact. Set*

$$\sup_{x \in \Omega^c} u(x) = a.$$

Then

$$u(x) \geqq a \quad \forall x \in \mathbf{R}^n.$$

In particular, according to [5] (*see also* [4]), $\Omega^c = \{u = a\}$ *is convex.*

Before we prove the theorem, we need several lemmas. Changing u by $u - a$, we can assume, without loss of generality, that $a = 0$.

Lemma 7. *Let $u \in \mathcal{P}$ and assume that Ω^c is a non-empty compact set. Furthermore assume*

$$\sup_{x \in \Omega^c} u(x) = 0. \tag{6}$$

Then for a suitable choice of the origin, for all $x \neq 0$ fixed, the function

$$r \longrightarrow \frac{u(rx)}{r^2}$$

is non-decreasing.

Proof. Denote by V the Newtonian potential of Ω^c, i.e.

$$V(x) = \int_{\Omega^c} \frac{c_n}{|x - y|^{n-2}} dy.$$

This is a bounded super-harmonic function in $\mathbf{R}^n$. Since V is harmonic in Ω, due to the maximum principle there is at least one point $\zeta_0 \in \Omega^c$ such that

$$V(\zeta_0) \geqq V(x), \quad \forall x \in \mathbf{R}^n.$$

Choose the origin at ζ_0.

Since

$$\Delta(u - V) = 1$$

in the sense of distributions and all second-order partial derivatives of $u - V$ are bounded harmonic functions, the Hessian of $u - V$ is a constant matrix, by Liouville's theorem.

Hence $u - V$ is a polynomial of degree two. Set

$$P(x) = u(x) - V(x) - u(0) + V(0).$$

Note that $|\nabla V(0)| = |\nabla u(0)| = 0$. Hence $P(0) = |\nabla P(0)| = 0$, and this implies that P is homogeneous.

Now consider the function

$$h(x) = x \cdot \nabla u(x) - 2u(x).$$

The function is continuous in $\mathbf{R}^n$, and for all $x \neq 0$ fixed,

$$\frac{d}{dr}\left(\frac{u(rx)}{r^2}\right) = \frac{1}{r^3}h(rx).$$

We will show that h is non-negative in $\mathbf{R}^n$. In fact

$$h(x) = -2u(x) \geqq 0 \quad \forall x \in \Omega^c .$$

On the other hand, by the homogeneity of P,

$$h(x) = x \cdot \nabla V(x) - 2V(x) + 2V(0) - 2u(0)$$

then

$$\lim_{|x|\to\infty} h(x) = 2V(0) - 2u(0) \geqq 0 .$$

Since h harmonic in Ω, by the minimum principle, h is positive in Ω. □

Corollary 1. *Under the hypothesis of Lemma 7, for all $\kappa \geqq 0$, the set $\{u \leqq \kappa\}$ is star-shaped with respect to the origin.*

Remark 4. The family $u_r(x) = u(rx)/r^2$ indexed by r is not relatively compact, since *a priori* $u(0) \neq 0$. Therefore, the monotonicity given by Lemma 7 does not mean that the blow-up at the origin is convergent.

Lemma 8. *Let $u \in \mathcal{P}$ and assume* (6). *Then any blow-up u_0 of u at $x_0 \in \partial\Lambda_0$, where $\Lambda_0 := \Omega^c \cap \{u = 0\}$, is a half-space solution.*

Before we prove this lemma, we need a result concerning a balanced energy functional, introduced by G.S. Weiss. We shall use a slightly different version of Weiss' formula. Define

$$\Phi(r, u, x_0) = r^{-n-2} \int_{B_r(x_0)} \left(|\nabla u|^2 + 2u\right) - r^{-n-3} \int_{\partial B_r(x_0)} 2u^2.$$

The following result is basically due to WEISS, see [10]. For the reader's convenience, we also give a proof.

Lemma 9 (Weiss). *Let $u \in \mathcal{P}$ and assume* (6). *Then for all $x_0 \in \mathbf{R}^n$, $\Phi(r, u, x_0)$ is non-decreasing with respect to r.*

Remark 5. The hypothesis (6) is crucial for the proof of Lemma 9. The lemma fails if we replace u by $u - u(x_0)$, unless $u(x_0) \geqq 0$. We can use this lemma in conjunction with blow-ups only when $x_0 \in \partial\Lambda_0$. Nevertheless, it is convenient to write $\Phi(r, u, x_0)$ in terms of

$$u_r(x) = \frac{u(x_0 + rx)}{r^2}, \qquad r > 0.$$

In that case

$$\Phi(r, u, x_0) = \int_{B_1(0)} \left(|\nabla u_r|^2 + 2u_r\right) - \int_{\partial B_1(0)} 2{u_r}^2 .$$

Proof of Lemma 9. We shall prove that the derivative of $\Phi(r, u, x_0)$ with respect to r is non-negative.

Indeed,

$$\Phi'(r) = \int_{B_1} \left(2\nabla u_r \cdot \nabla u_r{}' + 2u_r{}'\right) - \int_{\partial B_1} 4u_r u_r{}' ,$$

where

$$u_r{}'(x) = \frac{d}{dr} u_r(x) = \frac{1}{r} \left(\nabla u_r \cdot x - 2u_r\right).$$

Using integration by parts,

$$\int_{B_1} 2\nabla u_r \cdot \nabla u_r{}' = \int_{\partial B_1} 2(\nabla u_r \cdot \eta) u_r{}' - \int_{B_1} 2\Delta u_r u_r{}' .$$

Since

$$\nabla u_r \cdot x = r u_r{}'(x) + 2u_r$$

and $\eta = x$ on ∂B_1,

$$\Phi'(r) = \int_{\partial B_1} 2r(u_r{}')^2 + \int_{B_1} 2(1 - \chi_{\Omega_r}) u_r{}'.$$

The first integrand above is non-negative. The second one is also non-negative since we have assumed $u(x) \leqq 0$ for all $x \in \Omega^c$. □

Since $u_r{}' \equiv 0$ if and only if u is homogeneous of degree two, the above expression leads to the following important conclusion, already found in Weiss' paper for the obstacle problem.

Corollary 2 (Weiss). *Under the hypothesis of Lemma 9, the function*

$$v(x) = u(x_0 + x)$$

is homogeneous of degree two if and only if $\Phi(r, u, x_0)$ *is constant with respect to* r.

Remark 6. Let P be a degree-two homogeneous polynomial, with $\Delta P = 1$. Then $\Phi(r, P, 0)$ does not depend on r nor on P.

By this lemma we can set

$$\alpha_n = \Phi(r, P, 0) = \frac{1}{2} \int_{B_1} x_1^2.$$

where x_1 is the first coordinate component of $x \in B_1$. This is twice the value of $\Phi(r, U, 0)$, when U is a half-space solution,

$$\Phi(r, U, 0) = \frac{\alpha_n}{2}.$$

Remark 7. Let $u \in \mathcal{P}$ and $x_0 \in \partial\Omega$. Since by uniform convergence,

$$\Phi(r, u_0, 0) = \Phi(0^+, u, x_0) \leqq \Phi(r, u, x_0) \leqq \Phi(\infty^-, u, x_0) = \Phi(r, u_\infty, 0)$$

and any blow-up or blow-down is homogeneous, we are left with only three possibilities:

$$\Phi(0^+, u, x_0) = \Phi(\infty^-, u, x_0) = \frac{\alpha_n}{2},$$

or

$$\Phi(0^+, u, x_0) = \frac{\alpha_n}{2} \text{ and } \Phi(\infty^-, u, x_0) = \alpha_n,$$

or

$$\Phi(r, u_0, 0) = \Phi(\infty^-, u, x_0) = \alpha_n .$$

Proof of Lemma 8. If u_0 is a polynomial, then $\Phi(0^+, u, x_0) = \alpha_n$. Since α_n is the maximum value of $\Phi(r, u, x_0)$,

$$\Phi(r, u, x_0) = \alpha_n \quad \forall r > 0.$$

This means that u itself is a polynomial. This contradicts the assumption $\Omega^c \neq \emptyset$. □

Proof of Theorem 3. Recall from Lemma 7 that the origin is now fixed, and belongs to Ω^c. If $0 \in \partial\{u \leqq 0\}$, then $0 \in \partial\Lambda_0$. By Lemma 7 and Lemma 8 ,

$$\lim_{r\to 0} u_r(x) = \inf_{r>0} u_r(x) = u_0 \geqq 0.$$

The theorem is proved in this case.

If x_0 is an interior point of $\{u \leqq 0\}$, then by Corollary 1, the interior of $\{u \leqq 0\}$ is connected.

By Lemma 8 and Lemma 6, Λ_0 contains a truncated cone. Since u is subharmonic in the interior of $\{u \leqq 0\}$ and the interior of $\Lambda_0 \neq \emptyset$, by the maximum principle,

$$\{u \leqq 0\} = \{u = 0\}.$$

This completes the proof of the theorem. □

7. Global solutions with unbounded Ω^c

Theorem 4. *Let $u \in \mathcal{P}$ such that* $\text{int}(\Omega^c)$ *is non-empty and unbounded. Then, there is $a \in \mathbf{R}$ such that $u \geqq a$ and $\Omega^c = \{u = a\}$.*

In particular, according to [5] (*see also* [4]), *Ω^c is convex.*

Proof. Suppose that some blow-up u_∞ of u at infinity is a half-space solution. Then, by the third statement in Proposition 1, $u - u(x_0)$ is a half-space solution (modulo translation), $\forall x_0 \in \Omega^c$. And the theorem follows in this case.

Now, if no blow-up at infinity is a half-space solution, then by Lemma 3 we may assume that any blow-up u_∞ is a polynomial. The assumption $\text{int}(\Omega^c) \neq \emptyset$ prevents from u from being a polynomial, except in the trivial case $u \equiv a$.

Since Ω^c is unbounded, there exists a sequence $x^j \in \partial\Omega$ tending to ∞. In this case we may scale by $R_j = |x^j|$ so as to obtain, in the limit, a global solution with a free boundary $\tilde{x}$ on the unit sphere. By homogeneity then the ray $r\tilde{x}$ must lie in the free boundary. It thus follows that $D_e u_\infty \equiv 0$, for $e = \tilde{x}/|\tilde{x}|$. Hence

$$0 \leqq \varphi(r, e, u) \leqq \varphi(\infty, e, u) = \varphi(1, e, u_\infty) = 0,$$

and we conclude that $D_e u$ does not change sign for $e = \tilde{x}/|\tilde{x}|$, we assume, without lost of generality, $D_e u \geqq 0$ (otherwise we replace e by $-e$).

Now, for some x_0 with $B_r(x_0) \subset \{|\nabla u| = 0\}$, we can show that

$$\{x - se : \ x \in B_r(x_0), \ s > 0\} \subset \{|\nabla u| = 0\}.$$

In particular the sequence $u_m(x) := u(x - me) - u(x_0)$ is bounded (for any fixed x) and monotone. Thus it has a limit as m tends to infinity. It follows, moreover, that the limit function is independent of the e-direction. Therefore we will have an $(n-1)$-dimensional solution $\widehat{u}$.

First, suppose the lower-dimensional function $\widehat{u}$ is either a half-space solution, or falls into the hypotheses of Theorem 3. Then the lower-dimensional solution is convex and non-negative. Since $D_e u \geqq 0$ we conclude that $u \geqq 0$ (or more correctly $u(x) - u(x_0) \geqq 0$).

Due to the convexity of $\widehat{u}$, the positivity of u, and the fact that $D_e u \geqq 0$, we must have $\{|\nabla u| = 0\}$ connected. Hence we are reduced to the case of $u = u(x_0)$ in the set $\{|\nabla u| = 0\}$, and we can apply [6] to conclude that u is convex.

Next, if the lower-dimensional solution $\widehat{u}$ is neither of the above it must fall into the third category analyzed above. Hence we repeat our argument and translate $\widehat{u}$ again in a new direction and reduce the dimension further. Finally, by induction, we need to classify the one-dimensional solutions. However, the one-dimensional problem is solved by $x_1^2/2$, $(\max(0, x_1))^2/2$, or two separated solutions of the latter. Obviously any rotation of these are also possible solutions. And these all are non-negative solutions. □

8. Proof of Theorem 2

By the classification of global solutions, made in Sections 6 and 7, we have already proved that the global solutions of our problem coincide (in nature) with that of [6] and hence the proof of Theorem 2 follows the same pattern as that of Theorem III in [6].

The original result (see [4] or [5]) for non-negative solutions in the class $\mathcal{P}(c, 1, x_0)$, provides us with the following lemma.

Lemma 10. *Given a positive number ε, there exists t_ε such that if $u \in \mathcal{P}$ and $\delta_1(u) \geqq \varepsilon$, then in $B(0, t)$ ($\forall t < t_\varepsilon$) the boundary of Ω is the graph of a C^1 function (uniformly for the class) and u is (εt^2)-close to a half-space solution there.*

Now a simple proof of Lemma 10 can be given based on compactness, a contradictory argument, and the main result in [4]. Observe also that by classification of global solutions in Section 6 and 7, $u \in \mathcal{P}$ is non-negative. Therefore [4] applies.

Lemma 11. *Let* $\varepsilon, s > 0$*, and* $u \in \mathcal{P}(c, 1, x_0)$*. Assume that for some* $\nu_0 \in S^{n-1}$*,*

$$\sup_{x \in B_s(0)} \left| u(x_0 + x) - \frac{1}{2}[(x \cdot \nu_0)^+]^2 \right| \leqq \varepsilon s^2.$$

Then

$$s D_{\nu_0} u - |\nabla u|^2 \geqq 0 \quad \text{in } B_{s/2}(x_0),$$

provided ε *is small enough.*

The proof of the above lemma follows the same lines as that of the proof of Lemma 4.

For the following lemma, recall Remark 2, and the constant γ there, and suppose $0 < \varepsilon < \gamma/2$.

Lemma 12. *For* $0 < \varepsilon < \gamma/2$*, there exists* $\rho_\varepsilon > 0$ *such that if, for some* $\rho \leqq \rho_\varepsilon$ *and* $u \in \mathcal{P}(c, 1, x_0)$*,* $\delta_\rho(u) \geqq \varepsilon$*, then (for some direction* $\nu_0 = \nu_0(u)$*)*

$$x_0 + C(\rho, \tfrac{1}{3}, \nu_0) \subset \Omega^c \cap B_\rho(x_0) \subset \{u = u(x_0)\}, \tag{7}$$

and

$$u \geqq u(x_0) \qquad \text{in } B_\rho(x_0). \tag{8}$$

Proof. Let $\rho_1 \leqq t_\varepsilon/2$ (with t_ε as in Lemma 10), and suppose $\delta_{\rho_1}(u) \geqq \varepsilon$. Hence the hypothesis in Lemma 11 is fulfilled. In particular, we may use Lemmas 10 and 11, to observe that hypotheses in Lemmas 5 and 6 are fulfilled. Hence the first inclusion in (7) follows in a similar fashion to that in the proof of Lemmas 5 and 6. Here, however, we should notice the uniformity due to Lemmas 10 and 11, so that ρ_1 is independent of u.

Now the assumption $\delta_{\rho_1}(u, x_0) \geqq \varepsilon$ implies

$$\delta_{2\rho_1}(u, y) \geqq \varepsilon/2 \qquad \forall\, y \in \partial\{|\nabla u| > 0\} \cap B_{\rho_1}(x_0),$$

and an argument similar to that above applies. Hence we have

$$y + C(\rho_1, \tfrac{1}{3}, \nu_y) \in \Omega^c, \qquad \forall\, y \in \partial\{|\nabla u| > 0\} \cap B_{\rho_1}(x_0).$$

Since the opening of the cones are large enough they must overlap. This, in turn, implies that u is constant in Ω^c. Finally (8) follows by applying Lemma 11, and using (7). □

Now from the above lemma it follows that u does not change sign, the set $\{|\nabla u| = 0\}$ is connected and the free boundary $\partial\{|\nabla u| > 0\}$ is locally (in a uniform neighborhood, depending on the modulus of continuity $\sigma(r)$) a Lipschitz graph. The rest now follows as in [5], or [6].

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